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Random flights of massless particles and precursors

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Abstract. The diffusion of photons through a cloud of ionised plasma is considered. Phenomenological hyperbolic equations derived earlier from transient thermodynamics predict diffusion profiles which are typically double-peaked. The propagation of the diffusion front is causal and this gives rise to a precursor peak preceding the main diffusion peak. It is shown that this feature finds an explanation in terms of random walk theory. The configuration space for the displacements is taken to be Minkowski space and the random distributions are concentrated on the future light cone.

1. Introduction

The conventional theory of dissipative continuous media is based on equations of motion with parabolic characteristics and is, for this reason, afflicted with undesirable features such as infinite wavefront speeds for the transport of heat and viscous stresses. An extended and relativistic covariant theory, based on *hyperbolic* equations, was worked out by Israel (1976). The reader is also referred to Israel and Stewart (1979a, b, 1980). This new theory, called 'transient thermodynamics', predicts subluminal characteristic velocities and one expectes that it should be particularly accurate in describing the arrival of signal fronts.

The diffusion of photons through a radiative plasma is important in astrophysical problems. As was shown in an earlier paper (Schweizer 1984b, paper I), the general code of transient thermodynamics modifies the conventional diffusion equations just slightly: there is one single new term which guarantees that the diffusion front propagates causally. If the optical radius of the considered plasma is less than about twenty, the diffusion profiles tend to be *double peaked*: the conventional diffusion peak is preceded by a *precursor peak* which propagates at the characteristic velocity $1/\sqrt{3}$ times the velocity of light.

The appearance of the precursor has raised the question of whether this feature is nothing more than an artifact of the hyperbolic diffusion equations and has, in principle, nothing to do with reality. In addition, one needs to establish a lower bound for the range of optical thickness in which the phenomenological equations are applicable.

The analysis of this paper is based on random walk models for massless particles. In order to account fully for the relativistic character of these particles, the configuration space for the displacements is taken to be the future light cone in Minkowski space instead of the three-space.

For the case of one time plus one space dimension, random walk theory and transient thermodynamics provide *identical* results. The $(1+3)$-dimensional case is less immediate. However, a Green function derived in paper I from the hyperbolic diffusion equations follows easily from a random walk model, and for the case of quasi-stationary sources, the two theories provide identical results.

A numerical analysis of the diffusion equations shows that the profiles are rather insensitive to the detailed boundary conditions, provided the optical radius of the plasma is larger than about five. Together with the results derived in this paper one may, for this reason, conjecture that the transient thermodynamical diffusion equations are applicable down to optical radii as small as about five. But above all, it becomes clear that the precursor feature discussed in paper I is realistic and finds an explanation in terms of the particles which scatter into the forward direction only. Multiple-peaked luminosity profiles are common but ill understood in the context of observed X-ray transients. A thorough investigation of the hyperbolic diffusion equations and their implications is therefore worthwhile.

In § 2, we present the hyperbolic diffusion equations derived in paper I. Special consideration is given to the $(1+1)$-dimensional case. In § 3, we present a random walk model for massless particles in $(1+1)$ dimensions and show that the main result is identical with the corresponding one in § 2. In § 4, we present a random walk model for the $(1+3)$-dimensional case and discuss the overlap with transient thermodynamics as well as the range of applicability of the hyperbolic diffusion equations.

2. The hyperbolic diffusion approximation

The life-time τ of a photon in a low-density radiative plasma at high temperature is typically large; for instance, for a plasma of order $10^{-9}\,\mathrm{g\,cm^{-3}}$ and a temperature $T\sim 3\times 10^{7}$ K, one has $\tau\sim 10^{3}$ s; see table 2 in Schweizer (1984a). If for a given situation the typical diffusion time t_{D} is much shorter than τ, then the collisions of the photons with the electrons are basically elastic and coherent, i.e. one has, to lowest order, Thomson scattering only. This defines a photon diffusion process which depends on the electron number density n_{e} only, but not on the temperature or the equation of state of the plasma.

The general hyperbolic diffusion equations accounting for all the relativistic effects such as gravitational redshift and Doppler shift have been derived earlier in paper I. If the space–time is flat and the plasma has constant density the diffusion equations are given by

$$\partial N/\partial t+\nabla\cdot \boldsymbol{J}=S, \tag{1}$$

$$\kappa_{\mathrm{T}}\boldsymbol{J}+\partial \boldsymbol{J}/\partial t+\tfrac{1}{3}\nabla N=0. \tag{2}$$

The term $S(t,\boldsymbol{x})$ in the continuity equation (1) denotes a source of photons placed inside the plasma cloud, N is the photon number density, $\boldsymbol{J}$ is the photon number three-current, and $\kappa_{\mathrm{T}}=n_{\mathrm{e}}\sigma_{\mathrm{T}}$. The only but crucial modification due to transient thermodynamics is the $(\partial/\partial t)\boldsymbol{J}$ term in the transport equation (2). It is convenient to separate the variables N and $\boldsymbol{J}$ by differentiating (1) and (2) with respect to space and time. It follows that

$$\mathscr{L}N\equiv\partial N/\partial t+\partial^2 N/\partial t^2-\tfrac{1}{3}\Delta N=S+\partial S/\partial t, \tag{3}$$

$$\partial\boldsymbol{J}/\partial t+\partial^2\boldsymbol{J}/\partial t^2-\tfrac{1}{3}\nabla(\nabla\cdot\boldsymbol{J})=-\tfrac{1}{3}\nabla S. \tag{4}$$

Notice that we have *identified* $t \equiv t_T^{-1} t$ and $x \equiv \lambda_T^{-1} x$, where $\lambda_T = (n_e \sigma_T)^{-1}$. We split the photon number density into $N = N_0 + \delta N$, where N_0 is a stationary part satisfying equation (3) with $S = 0$. At the boundary of the plasma, the distribution function $P(t)$ of the photons as a function of their time of escape from the cloud is proportional to δN. It is, for this reason, sufficient to analyse equation (3) for δN.

As shown in paper I, the diffusion profiles associated with (3) are typically double-peaked: the diffusion front reaches the boundary of the cloud with a precursor peak at the precursor time $\sqrt{3}\ R$, where R denotes the optical radius. The main diffusion peak arrives later at the diffusion time $t_D \simeq 0.35\,R^2$. Quantitative results and some immediate implications with respect to the luminosity profiles of X-ray bursters have been discussed in paper I. The concern of this paper is to understand this precursor feature in terms of a consistent random walk model.

To start with, we consider the restricted problem with one time and one space dimension. After rescaling the spatial variable by $\sqrt{3}$, equation (3) simplifies to

$$LN \equiv \partial N/\partial t + \partial^2 N/\partial t^2 - \partial^2 N/\partial x^2 = S + \partial S/\partial t. \tag{5}$$

The Green function G satisfying the equation

$$LG = \delta(x)\delta(t) \tag{6}$$

can be given in closed form as

$$G(t, x) = \tfrac{1}{2}\theta(t - |x|)\, e^{-t/2} I_0[\tfrac{1}{2}(t^2 - x^2)^{1/2}]. \tag{7}$$

The solution of (5) is the convolution integral

$$N = G * (S + \partial S/\partial t). \tag{8}$$

We define the *effective* Green function G_{eff} by

$$G_{\text{eff}} * S = G * (S + \partial S/\partial t) \tag{9}$$

or in other words

$$\begin{aligned} G_{\text{eff}} &= (1 + \partial/\partial t)G \\ &= \tfrac{1}{2}\delta(t - |x|)\, e^{-t/2} + \tfrac{1}{4}\theta(t - |x|)\, e^{-t/2} \\ &\quad \times [I_0[\tfrac{1}{2}(t^2 - x^2)^{1/2}] + t\, I_1[\tfrac{1}{2}(t^2 - x^2)^{1/2}]/(t^2 - x^2)^{1/2}]. \end{aligned} \tag{10}$$

The symbols I_0 and I_1 denote the modified Bessel functions of the first kind of order 0 and 1, respectively. The first terms of the power series expansion of the analytical function in the square brackets in (10) are as follows:

$$I_0[\tfrac{1}{2}(t^2 - x^2)^{1/2}] + t\frac{I_1[\tfrac{1}{2}(t^2 - x^2)^{1/2}]}{(t^2 - x^2)^{1/2}} = 1 + \frac{t}{4} + \frac{t^2}{16} - \frac{x^2}{16} \cdots \tag{11}$$

The $(\partial/\partial t)S$ term in (5) does not appear in the conventional diffusion equation. As we can tell from expression (10), this term is responsible for the precursor-type part $\tfrac{1}{2}\delta(t - |x|)\exp(-\tfrac{1}{2}t)$ which accounts for all the photons which scatter into the forward direction only, i.e. move always on the initial light cone. It is obvious from this consideration that the conventional non-causal treatment of photon diffusion is *a priori* unable to provide precursors or to describe the arrival of the diffusion front properly.

3. Random flights of massless particles in (1+1) dimensions

The description of random flights of massless particles requires special care: the particle velocity during a displacement is always equal to the speed of light $c=1$, i.e. all the displacements occur on the light cone. The configuration space for the displacements should, for this reason, not be $\mathbb{R}^3$ but rather Minkowski space, and the random distribution $\tau(t, x)$ should be concentrated on the future light cone.

We define the following random distribution on the $(1+1)$-dimensional Minkowski space describing a photon propagating on the light cone and having a mean free path of order unity:

$$\tau(t, x)=\tfrac{1}{2}\delta(t-|x|)\,\mathrm{e}^{-|x|}. \tag{12}$$

For a given space–time point (t, x), the number $\tau(t, x)$ is equal to the probability $W_1(t, x)$ that the particle released at the origin is at (t, x) after one displacement. Notice that the space–time integral of τ is normalised to unity. The probability $W_2(t, x)$ that the particle is at (t, x) after two displacements is equal to the convolution integral $\tau * \tau$, and more generally

$$W_N(t, x)=\underbrace{\tau * \tau * \ldots * \tau}_{N \text{ factors}}. \tag{13}$$

For an introductory discussion of Markoff's method for random walk problems, the reader is referred to Chandrasekhar (1943).

It is straightforward to compute W_1, W_2, W_3, W_4, etc directly from (13). For $t>0$, the following identity proves to be helpful:

$$\theta(t-|x'|-|x-x'|)=\theta(t-|x|)\theta(\tfrac{1}{2}(t+x)-x')\theta(\tfrac{1}{2}(t-x)+x'). \tag{14}$$

One finds that

$$\begin{aligned}
W_1&=\tfrac{1}{2}\delta(t-|x|)\,\mathrm{e}^{-t},\\
W_2&=\tfrac{1}{2}\delta(t-|x|)\,\mathrm{e}^{-t}(\tfrac{1}{2}t)+\tfrac{1}{4}\theta(t-|x|)\,\mathrm{e}^{-t},\\
W_3&=\tfrac{1}{2}\delta(t-|x|)\,\mathrm{e}^{-t}(1/2!)(\tfrac{1}{2}t)^2+\tfrac{1}{4}\theta(t-|x|)\,\mathrm{e}^{-t}\tfrac{3}{4}t,\\
W_4&=\tfrac{1}{2}\delta(t-|x|)\,\mathrm{e}^{-t}(1/3!)(\tfrac{1}{2}t)^3+\tfrac{1}{4}\theta(t-|x|)\,\mathrm{e}^{-t}(\tfrac{5}{16}t^2-\tfrac{1}{16}x^2),
\end{aligned}$$

and so forth.

The probability $W(t, x)$ that the particle is at (t, x) *independently* of the number of displacements is given by the sum over all the probabilities W_N, i.e.

$$\begin{aligned}
W(t, x)&=\sum_{N=1}^{\infty} W_N(t, x)\\
&=\tfrac{1}{2}\delta(t-|x|)\,\mathrm{e}^{-t/2}+\tfrac{1}{4}\theta(t-|x|)(1+\tfrac{1}{4}t+\tfrac{1}{16}t^2-\tfrac{1}{16}x^2\ldots)\,\mathrm{e}^{-t/2}.
\end{aligned} \tag{16}$$

A comparison of (16) with expression (10) and the expansion (11) suggests that

$$W(t, x)=G_{\mathrm{eff}}(t, x). \tag{17}$$

The proof of this goes as follows. The Fourier transform of τ is given by

$$\tilde{\tau}(\omega, p)=\int \mathrm{d}x \int \mathrm{d}t\, \mathrm{e}^{\mathrm{i}px}\, \mathrm{e}^{\mathrm{i}\omega t}\tau(t, x)=\frac{1-\mathrm{i}\omega}{(1-\mathrm{i}\omega)^2+p^2}. \tag{18}$$

The probability distribution $W_N(t, x)$ is (see (13))

$$W_N(t,x) = (2\pi)^{-2} \int \mathrm{d}\omega \int \mathrm{d}p \, \mathrm{e}^{-\mathrm{i}px} \, \mathrm{e}^{-\mathrm{i}\omega t} [\tilde{\tau}(\omega, p)]^N. \tag{19}$$

The sum over all W_N leads to the geometric series

$$\sum_{N=1}^{\infty} \left(\frac{1-\mathrm{i}\omega}{(1-\mathrm{i}\omega)^2 + p^2} \right)^N = \frac{1-\mathrm{i}\omega}{p^2 - \omega^2 - \mathrm{i}\omega}. \tag{20}$$

This yields

$$W(t,x) = (1 + \partial/\partial t) w(t,x) \tag{21}$$

where

$$w(t,x) \equiv \frac{1}{(2\pi)^2} \int \mathrm{d}\omega \int \mathrm{d}p \frac{\mathrm{e}^{-\mathrm{i}px} \, \mathrm{e}^{-\mathrm{i}\omega t}}{p^2 - \omega^2 - \mathrm{i}\omega}. \tag{22}$$

The integral (22) is, in fact, the Fourier representation for $G(t, x)$, or in other words

$$W(t,x) = (1 + \partial/\partial t) G(t,x) = G_{\mathrm{eff}}(t,x). \tag{23}$$

This completes the proof. For t fixed, $W(t, x)$ is a probability distribution on the space-like surface $t = \text{constant}$. Since the particle reaches this surface after a finite number of displacements, the integral of $W(t, x)$ over this space-like surface is equal to unity.

The advantages of this relativistic treatment are obvious: it provides a proper description of the causal properties of the random walk process and leads automatically to the probability distribution independent of N. The phenomenological diffusion approximation is, in this case, exact and there is no need to go to the limit of large N. In non-relativistic treatments, the transition from the discrete parameter N to a continuous time t is, in general, not defined for small N, since the integrand in the Fourier transform of W_N is the Nth power of an oscillating function. All these difficulties disappear in the relativistic approach presented here.

4. The (1+3)-dimensional case

The obvious generalisation of the random distribution (12) to (1+3) dimensions is

$$\tau(t, \boldsymbol{x}) = (1/4\pi) \delta(t-r) \mathrm{e}^{-r}/r. \tag{24}$$

In order to compute $W(t, \boldsymbol{x})$, we start with the Fourier transform

$$\tilde{\tau}(\omega, \boldsymbol{p}) = \int \mathrm{d}t \int \mathrm{d}^3x \, \mathrm{e}^{\mathrm{i}\omega t} \, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \tau(t, \boldsymbol{x}) = [(1-\mathrm{i}\omega)^2 + p^2]^{-1}. \tag{25}$$

This leads to the geometric series

$$\sum_{N=1}^{\infty} \left(\frac{1}{(1-\mathrm{i}\omega)^2 + p^2} \right)^N = \frac{1}{p^2 - \omega^2 - 2\mathrm{i}\omega}. \tag{26}$$

The resulting probability distribution

$$W(t, \boldsymbol{x}) = \frac{1}{(2\pi)^4} \int \mathrm{d}\omega \int \mathrm{d}^3p \frac{\mathrm{e}^{-\mathrm{i}\omega t} \, \mathrm{e}^{-\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}}{p^2 - \omega^2 - 2\mathrm{i}\omega} \tag{27}$$

is the Fourier representation of the Green function associated with the operator

$$l \equiv 2\,\partial/\partial t + \partial^2/\partial t^2 - \Delta. \tag{28}$$

The poles of the integrand in (27) are in the lower half of the complex plane. It follows that

$$W(t, x) = \frac{2}{(2\pi)^2}\,\theta(t)\,\frac{e^{-t}}{r}\int_0^\infty dp\,p\,\sin(pr)\,\frac{\sin[t(p^2-1)^{1/2}]}{(p^2-1)^{1/2}}. \tag{29}$$

Making use of the integral 3.876 in Gradstein and Ryzhik (1981), one can give $W(t, x)$ in closed form as follows:

$$W(t, x) = \frac{1}{4\pi}\,\delta(t-r)\,\frac{e^{-t}}{r} + \frac{1}{4\pi}\,e^{-t}\theta(t-r)\,\frac{I_1[(t^2-p^2)^{1/2}]}{(t^2-p^2)^{1/2}}. \tag{30}$$

If we rescale times and lengths everywhere according to the prescription

$$t' = 2t, \qquad x' = 2/3x \tag{31}$$

the Green function (30) turns into

$$W(t', x') = \frac{\sqrt{3}}{4\pi}\,\delta\left(\frac{t'}{\sqrt{3}} - r'\right)\frac{e^{-t'/2}}{r} + \frac{3}{8\pi}\,e^{-t'/2}\theta\left(\frac{t'}{\sqrt{3}} - r'\right)\frac{I_1[\frac{1}{2}\sqrt{3}(\frac{1}{3}t'^2 - r'^2)^{1/2}]}{(\frac{1}{3}t'^2 - r'^2)^{1/2}}. \tag{32}$$

This is the Green function associated with the linear operator $\mathscr{L}$ in (3). This (1 + 3)-dimensional result is different from the corresponding (1 + 1)-dimensional one in as much as it does not provide G_{eff} but G. In other words, the diffusion process defined by the random distribution (24) does not cover the $(\partial/\partial t)S$ term in (3). This is, however, expected! Let us explain this in some more detail.

(i) At small optical depth $R \leqslant 1$, the observed characteristic velocity of the diffusion front depends on the time variation of the source S. For sources varying little over one mean collision time t_T, the photon number density δN is everywhere dominated by photons which have scattered with the electrons a couple of times already. Consequently, the effective velocity of the diffusion front is $1/\sqrt{3}$. If the source is highly transient, i.e. if most of the photons are released during a time period shorter than one mean free collision time, then the photon number density δN at $R \leqslant 1$ is dominated by *unscattered* photons just released from the source. The effective velocity of the diffusion front is, in this case, equal to the speed of light. At larger and larger optical radii $R > 1$, the photon number density δN is more and more dominated by scattered photons and the effective characteristic velocity decreases to $1/\sqrt{3}$.

(ii) In the (1 + 1)-dimensional case, the effective speed of the diffusion front is obviously always equal to the speed of light. This is, in fact, the reason that the random distribution (12) and transient thermodynamics provide identical results.

(iii) The random distribution (24) is good for the description of diffusion processes with *constant* characteristic velocities. For *quasi-stationary* sources, $S \gg (\partial/\partial t)S$. After rescaling times and lengths according to the prescription (31), the random distribution (24) provides the same diffusion process like equation (3). Conversely, for highly transient sources the characteristic velocity at $R \leqslant 1$ is equal to the speed of light. Equation (3) is positively not applicable to this case. Random walk theory provides, on the other hand, the equation $lW = S$, where l is the operator (28). The characteristic velocity according to this equation is, as required, equal to the speed of light.

(iv) For highly transient sources, the diffusion process defined by the random distribution (24) is not applicable to the cases $R > 1$, since it does not account for the decrease of the effective characteristic velocity due to the increase in optical depth. The transient thermodynamical diffusion equation (3) is, by contrast, sensitive to the effects caused by the rapid variation of S. A numerical analysis shows that for optical radii $R > 5$ the particular boundary conditions one chooses have little impact on the surface diffusion profile. And the diffusion front is anyway insensitive to the boundary conditions. This suggests that the transient thermodynamical diffusion equations are good for the description of cases $R > 5$. It remains, however, to be investigated what happens in the transition zone $1 < R < 5$, and how the $(\partial/\partial t)S$ term can be established consistently from random walk theory.

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